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RULED MINIMAL LAGRANGIAN SUBMANIFOLDS OF COMPLEX PROJECTIVE 3-SPACE*

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Abstract. We show how a ruled minimal Lagrangian submanifold of complex projective 3-space may be used to construct two related minimal surfaces in the 5-sphere.

Key words. Complex projective space, Lagrangian submanifold, sphere, minimal surface

AMS subject classifications. 53B25, 53B20

1. Introduction. In previous papers [1], [2] we showed how a Lagrangian submanifold M of complex projective 3-space $\mathbb{C}P^3(4)$ satisfying Chen's equality [7] but having no totally geodesic points may be used to construct a minimal surface in the unit 5-sphere $S^5(1)$ with ellipse of curvature a circle.

In this paper, we replace the assumption concerning Chen's equality with the assumption that M is minimal and admits a foliation by asymptotic curves, that is to say curves with vanishing normal curvature. In fact, these curves turn out to be geodesics of $\mathbb{C}P^3(4)$ (hence our description of M as a *ruled* submanifold of $\mathbb{C}P^3(4)$), and we show that the local construction referred to above may be applied to M to give two minimal surfaces in $S^5(1)$ whose ellipses of curvature are not circles. We also show that these minimal surfaces are related by a transform which generalises that of the polar (see [3], [9]) for linearly full minimal surfaces in $S^5(1)$ whose ellipses of curvature are circles. In a forthcoming paper [4], we will show that this transform may be defined for all non totally geodesic minimal surfaces in $S^5(1)$.

2. Ruled minimal Lagrangian submanifolds. Let M be a Lagrangian submanifold of $\mathbb{C}P^3(4)$. That is to say, if J is the complex structure of $\mathbb{C}P^3(4)$, then J maps the tangent bundle of M onto the normal bundle. Let $\tilde{\nabla}$ denote the Riemannian connection on $\mathbb{C}P^3(4)$, and ∇ , ∇^\perp the induced connections on M and the normal bundle of M . Let $h(X, Y) = \tilde{\nabla}_X Y - \nabla_X Y$ denote the second fundamental form of M , and, if N is a normal vector field, let $A_N(X) = -\tilde{\nabla}_X N + \nabla_X^\perp N$ denote the corresponding shape operator. If $\langle \cdot, \cdot \rangle$ denotes the Fubini-Study metric on $\mathbb{C}P^3(4)$, then [5, 8], the cubic form

$$C(X, Y, Z) = \langle h(X, Y), JZ \rangle = \langle A_{JZ}(X), Y \rangle \quad (1)$$

is symmetric in X , Y and Z . In particular,

$$A_{JX}(Y) = A_{JY}(X) = -Jh(X, Y). \quad (2)$$

We now assume that M admits a smooth unit length vector field \mathbf{e}_1 whose integral curves are asymptotic curves in M , that is to say they have zero normal curvature, so that

$$h(\mathbf{e}_1, \mathbf{e}_1) = 0. \quad (3)$$

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If $A_{J\mathbf{e}_1}$ vanishes identically at some point $p \in M$, then M satisfies Chen's equality at p (see [7]), and the situation in which this holds on an open subset of M has been discussed in [1] and [2]. Since we are dealing with a local theory here, we will from now on assume that M does not satisfy Chen's equality at any point.

It follows from (2) and (3) that $A_{J\mathbf{e}_1}\mathbf{e}_1 = 0$, so we may choose eigenvectors \mathbf{e}_2 and \mathbf{e}_3 of $A_{J\mathbf{e}_1}$ such that $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is an orthonormal basis of the tangent space of M . Let λ_2, λ_3 be the eigenvalues corresponding to $\mathbf{e}_2, \mathbf{e}_3$ respectively.

We now assume that M is minimal, so that

$$0 = \langle h(\mathbf{e}_2, \mathbf{e}_2) + h(\mathbf{e}_3, \mathbf{e}_3), J\mathbf{e}_1 \rangle = \langle A_{J\mathbf{e}_1}\mathbf{e}_2, \mathbf{e}_2 \rangle + \langle A_{J\mathbf{e}_1}\mathbf{e}_3, \mathbf{e}_3 \rangle = \lambda_2 + \lambda_3. \quad (4)$$

Thus $\lambda_2 = -\lambda_3 = \lambda$, where we may assume that λ is a strictly positive function on M and $\mathbf{e}_2, \mathbf{e}_3$ are smooth unit vector fields.

If we put $a = \langle A_{J\mathbf{e}_2}\mathbf{e}_2, \mathbf{e}_2 \rangle$, $b = \langle A_{J\mathbf{e}_2}\mathbf{e}_2, \mathbf{e}_3 \rangle$ then it is easy to check using (2), (3) and (4) that, with respect to the orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, we have the following matrix expressions.

$$A_{J\mathbf{e}_1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & -\lambda \end{pmatrix}, \quad (5)$$

$$A_{J\mathbf{e}_2} = \begin{pmatrix} 0 & \lambda & 0 \\ \lambda & a & b \\ 0 & b & -a \end{pmatrix}, \quad (6)$$

$$A_{J\mathbf{e}_3} = \begin{pmatrix} 0 & 0 & -\lambda \\ 0 & b & -a \\ -\lambda & -a & -b \end{pmatrix}. \quad (7)$$

Let z_j^i be the connection 1-forms on M defined by

$$\nabla \mathbf{e}_j = z_j^i \mathbf{e}_i, \quad (8)$$

and define the connection coefficients z_{kj}^i by

$$z_j^i(\mathbf{e}_k) = z_{kj}^i, \quad (9)$$

so that

$$z_{kj}^i = \langle \nabla_{\mathbf{e}_k} \mathbf{e}_j, \mathbf{e}_i \rangle = -z_{ki}^j. \quad (10)$$

We use the fundamental equations of submanifold theory, namely the Gauss, Codazzi and Ricci equations, to find relations between z_{jk}^i , a , b and λ . However, for Lagrangian submanifolds, the Gauss and Ricci equations are equivalent.

First consider the Codazzi equations, namely,

$$\nabla_X^\perp (h(Y, Z)) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z) = \nabla_Y^\perp (h(X, Z)) - h(\nabla_Y X, Z) - h(X, \nabla_Y Z).$$

If we apply J to this expression and take (2) into account, we see that the Codazzi equations are equivalent to

$$\nabla_X (A_{JY}Z) - A_{J(\nabla_X Y)}Z - A_{JY}(\nabla_X Z) = \nabla_Y (A_{JX}Z) - A_{J(\nabla_Y X)}Z - A_{JX}(\nabla_Y Z). \quad (11)$$

Equations (5), (6) and (7) may be used to show that (11) is equivalent to the following system (12)-(19).

$$z_{11}^2 = z_{11}^3 = 0, \quad z_{12}^3 = z_{21}^3 = -z_{31}^2, \quad z_{21}^2 = z_{31}^3, \quad (12)$$

$$\mathbf{e}_1(\lambda) = -2z_{21}^2\lambda, \quad (13)$$

$$\mathbf{e}_2(\lambda) = -2z_{32}^3\lambda, \quad (14)$$

$$\mathbf{e}_3(\lambda) = 2z_{22}^3\lambda, \quad (15)$$

$$\mathbf{e}_1(a) = 2bz_{12}^3 - az_{21}^2 - 2z_{32}^3\lambda, \quad (16)$$

$$\mathbf{e}_1(b) = -2az_{12}^3 - bz_{21}^2 + 2z_{22}^3\lambda, \quad (17)$$

$$\mathbf{e}_3(a) - \mathbf{e}_2(b) = 3az_{22}^3 + 3bz_{32}^3 + 4z_{12}^3\lambda, \quad (18)$$

$$\mathbf{e}_3(b) + \mathbf{e}_2(a) = 3bz_{22}^3 - 3az_{32}^3 - 2z_{21}^2\lambda. \quad (19)$$

In particular, we note from (12) that $\nabla_{\mathbf{e}_1}\mathbf{e}_1 = 0$, so that, by (3), the integral curves of \mathbf{e}_1 are geodesics in $CP^3(4)$. We have thus proved the following lemma.

LEMMA 1. *Let M be a minimal Lagrangian submanifold of $CP^3(4)$. If M admits a foliation by asymptotic curves then these curves are geodesics of $CP^3(4)$, so that M is a ruled submanifold.*

We next investigate the Gauss curvature equation, which states that the curvature tensor R of ∇ is given by

$$R(X, Y)Z = \langle Y, Z \rangle X - \langle X, Z \rangle Y + (A_{h(Y, Z)}X - A_{h(X, Z)}Y).$$

Taking (2) into account, the above equation is equivalent to

$$R(X, Y)Z = \langle Y, Z \rangle X - \langle X, Z \rangle Y + [A_{JX}, A_{JY}]Z. \quad (20)$$

Using (5), (6), (7) and (12), we find that (20) is equivalent to the following system (21)-(24).

$$\mathbf{e}_1(z_{12}^3) = -2z_{12}^3z_{21}^2, \quad \mathbf{e}_1(z_{21}^2) = -1 + (z_{12}^3)^2 - (z_{21}^2)^2 + \lambda^2, \quad (21)$$

$$\mathbf{e}_1(z_{22}^3) = -\mathbf{e}_3(z_{21}^2) - z_{21}^2z_{22}^3, \quad \mathbf{e}_1(z_{32}^3) = \mathbf{e}_2(z_{21}^2) - z_{21}^2z_{32}^3, \quad (22)$$

$$\mathbf{e}_2(z_{12}^3) = -\mathbf{e}_3(z_{21}^2) + 2b\lambda, \quad \mathbf{e}_3(z_{12}^3) = \mathbf{e}_2(z_{21}^2) - 2a\lambda, \quad (23)$$

$$\mathbf{e}_2(z_{32}^3) = \mathbf{e}_3(z_{22}^3) + 2a^2 + 2b^2 - 1 - 3(z_{12}^3)^2 - (z_{21}^2)^2 - (z_{22}^3)^2 - (z_{32}^3)^2 + \lambda^2. \quad (24)$$

The Gauss, Codazzi and Ricci equations provide a full set of integrability conditions, so we have the following theorem.

THEOREM 1. *Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be an orthonormal moving frame on a simply-connected Riemannian manifold M , and let $\{z_{kj}^i\}$ be the connection coefficients of the corresponding Riemannian connection. If there exist functions $\lambda > 0$, a , b on M satisfying (12)-(19) and (21)-(24) then M may be isometrically immersed as a ruled minimal Lagrangian submanifold of $\mathbb{CP}^3(4)$ with shape operator A given by (5), (6) and (7). Moreover the immersion is unique up to holomorphic isometries of $\mathbb{CP}^3(4)$.*

We now show the existence of such submanifolds M of $\mathbb{CP}^3(4)$. In fact, we will show in a forthcoming paper [4] that a solution $f(x, y)$ to the sinh-Gordon equation

$$f_{xx} + f_{yy} + 4 \sinh f = 0$$

determines a ruled minimal Lagrangian submanifold of $\mathbb{CP}^3(4)$ with the property that the distribution orthogonal to the rulings is integrable. Indeed, let f be such a function and let $\mu(t, x, y) = \cos t \sinh f + \cosh f$. Then define a Riemannian metric on a suitable open subset of \mathbb{R}^3 by taking $\mathbf{e}_1 = -2(\partial/\partial t)$, $\mathbf{e}_2 = \mu^{-1/2}(\partial/\partial x)$, $\mathbf{e}_3 = \mu^{-1/2}(\partial/\partial y)$ to be an orthonormal moving frame. It follows easily from the Koszul formula that the non-zero connection coefficients of the corresponding Riemannian connection are given by

$$z_{21}^2 = -z_{22}^1 = z_{31}^3 = -z_{33}^1 = -\frac{\mu_t}{\mu}, \quad z_{32}^3 = -z_{23}^2 = \frac{\mu_x}{2\mu^{3/2}}, \quad z_{22}^3 = -z_{23}^2 = -\frac{\mu_y}{2\mu^{3/2}},$$

and, in particular, $\{\mathbf{e}_2, \mathbf{e}_3\}$ span an integrable distribution. Then taking

$$\lambda = 1/\mu, \quad a = \frac{f_x \sin t}{2\mu^{3/2}}, \quad b = \frac{f_y \sin t}{2\mu^{3/2}},$$

it may be checked that (12)-(19) and (21)-(24) are all satisfied, so we may apply Theorem 1 to prove the existence of a corresponding ruled minimal Lagrangian submanifold of $\mathbb{CP}^3(4)$.

Returning now to the general situation, let \mathbf{E}_0 be a local horizontal lift of a ruled minimal Lagrangian submanifold M of $\mathbb{CP}^3(4)$ to the total space of the Hopf fibration $\pi : S^7(1) \rightarrow \mathbb{CP}^3(4)$, where $S^7(1)$ is the unit sphere in $\mathbb{R}^8 = \mathbb{C}^4$. The existence of such a lift follows from a result of Reckziegel [10], and any two such lifts \mathbf{E}_0 $\tilde{\mathbf{E}}_0$ are related by $\tilde{\mathbf{E}}_0 = e^{i\theta} \mathbf{E}_0$, where θ is a constant.

For $j = 1, 2, 3$, let \mathbf{E}_j be the image under the derivative $d\mathbf{E}_0$ of \mathbf{e}_j , and let $\mathcal{E} = (\mathbf{E}_0, \mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3)$ be the map from M to the unitary group $U(4)$ so constructed.

We now write down the moving frame equations of \mathcal{E} . In fact, if $\omega_1, \omega_2, \omega_3$ is the dual frame to $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$, a routine calculation using (5)-(9) and (12) shows that

$$d\mathcal{E} = \mathcal{E}(\alpha + i\beta) \tag{25}$$

where

$$\alpha = \begin{pmatrix} 0 & -\omega_1 & -\omega_2 & -\omega_3 \\ \omega_1 & 0 & -z_{21}^2\omega_2 + z_{12}^3\omega_3 & -z_{12}^3\omega_2 - z_{21}^2\omega_3 \\ \omega_2 & z_{21}^2\omega_2 - z_{12}^3\omega_3 & 0 & -z_{12}^3\omega_1 - z_{22}^3\omega_2 - z_{32}^3\omega_3 \\ \omega_3 & z_{12}^3\omega_2 + z_{21}^2\omega_3 & z_{12}^3\omega_1 + z_{22}^3\omega_2 + z_{32}^3\omega_3 & 0 \end{pmatrix} \tag{26}$$

and

$$\beta = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda\omega_2 & -\lambda\omega_3 \\ 0 & \lambda\omega_2 & \lambda\omega_1 + a\omega_2 + b\omega_3 & b\omega_2 - a\omega_3 \\ 0 & -\lambda\omega_3 & b\omega_2 - a\omega_3 & -\lambda\omega_1 - a\omega_2 - b\omega_3 \end{pmatrix}. \quad (27)$$

Note that taking a different horizontal lift \mathbf{E}_0 would imply that we multiply \mathbf{E}_0 (and thus also \mathbf{E}_1 , \mathbf{E}_2 and \mathbf{E}_3) by a factor $e^{i\theta}$, where θ is a constant. Thus we may choose a lift for which \mathcal{E} lies in $SU(4)$ at some point. It then follows from (26) and (27) that \mathcal{E} always lies in $SU(4)$ so, by choosing a suitable horizontal lift E_0 , we may assume that

$$\mathcal{E} : M \rightarrow SU(4). \quad (28)$$

We now compose \mathcal{E} with a suitably chosen standard double-cover of $SO(6)$ by $SU(4)$ to obtain a map $\mathcal{U} : M \rightarrow SO(6)$. In fact, if we let V be the 6-dimensional real subspace of the second exterior power $\wedge^2 \mathbb{C}^4$ of \mathbb{C}^4 spanned by

$$\mathbf{U}_1 = \frac{1}{\sqrt{2}}(\mathbf{E}_0 \wedge \mathbf{E}_3 + \mathbf{E}_1 \wedge \mathbf{E}_2), \quad \mathbf{U}_2 = \frac{1}{\sqrt{2}}(\mathbf{E}_0 \wedge \mathbf{E}_1 + \mathbf{E}_2 \wedge \mathbf{E}_3), \quad (29)$$

$$\mathbf{U}_3 = \frac{1}{\sqrt{2}}(\mathbf{E}_0 \wedge \mathbf{E}_2 + \mathbf{E}_3 \wedge \mathbf{E}_1), \quad \mathbf{U}_4 = \frac{1}{i\sqrt{2}}(\mathbf{E}_0 \wedge \mathbf{E}_1 - \mathbf{E}_2 \wedge \mathbf{E}_3), \quad (30)$$

$$\mathbf{U}_5 = \frac{1}{i\sqrt{2}}(\mathbf{E}_0 \wedge \mathbf{E}_2 - \mathbf{E}_3 \wedge \mathbf{E}_1), \quad \mathbf{U}_6 = \frac{1}{i\sqrt{2}}(\mathbf{E}_0 \wedge \mathbf{E}_3 - \mathbf{E}_1 \wedge \mathbf{E}_2), \quad (31)$$

then V is a constant subspace. If we extend the standard inner product on \mathbb{C}^4 to $\wedge^2 \mathbb{C}^4$ and identify V with \mathbb{E}^6 by choosing an orthonormal basis of V , then we obtain our required map $\mathcal{U} = (U_1, \dots, U_6) : M \rightarrow SO(6)$.

We now write down the moving frame equations of \mathcal{U} . In fact, if

$$d\mathcal{U} = \mathcal{U}\Omega \quad (32)$$

for a 6×6 matrix Ω of 1-forms on M , then a calculation using (26) and (27) shows that

$$\Omega = \begin{pmatrix} 0 & (z_{12}^3 - 1)\omega_2 + z_{21}^2\omega_3 & (z_{12}^3 + 1)\omega_1 + z_{22}^3\omega_2 + z_{32}^3\omega_3 \\ (1 - z_{12}^3)\omega_2 - z_{21}^2\omega_3 & 0 & -z_{21}^2\omega_2 + (z_{12}^3 - 1)\omega_3 \\ -(z_{12}^3 + 1)\omega_1 - z_{22}^3\omega_2 - z_{32}^3\omega_3 & z_{21}^2\omega_2 + (1 - z_{12}^3)\omega_3 & 0 \\ \lambda\omega_3 & 0 & -\lambda\omega_2 \\ -b\omega_2 + a\omega_3 & -\lambda\omega_2 & -\lambda\omega_1 - a\omega_2 - b\omega_3 \\ \lambda\omega_1 + a\omega_2 + b\omega_3 & \lambda\omega_3 & -b\omega_2 + a\omega_3 \end{pmatrix} \quad (33)$$

$$\begin{pmatrix} -\lambda\omega_3 & b\omega_2 - a\omega_3 & -\lambda\omega_1 - a\omega_2 - b\omega_3 \\ 0 & \lambda\omega_2 & -\lambda\omega_3 \\ \lambda\omega_2 & \lambda\omega_1 + a\omega_2 + b\omega_3 & b\omega_2 - a\omega_3 \\ 0 & -z_{21}^2\omega_2 + (z_{12}^3 + 1)\omega_3 & -(z_{12}^3 + 1)\omega_2 - z_{21}^2\omega_3 \\ z_{21}^2\omega_2 - (z_{12}^3 + 1)\omega_3 & 0 & (1 - z_{12}^3)\omega_1 - z_{22}^3\omega_2 - z_{32}^3\omega_3 \\ (z_{12}^3 + 1)\omega_2 + z_{21}^2\omega_3 & (z_{12}^3 - 1)\omega_1 + z_{22}^3\omega_2 + z_{32}^3\omega_3 & 0 \end{pmatrix}.$$

It is clear from the above that $d\mathbf{U}_2(\mathbf{e}_1) = 0$, while

$$d\mathbf{U}_2(\mathbf{e}_2) = (-1 + z_{12}^3)\mathbf{U}_1 + z_{21}^2\mathbf{U}_3 - \lambda\mathbf{U}_5, \quad (34)$$

$$d\mathbf{U}_2(\mathbf{e}_3) = z_{21}^2\mathbf{U}_1 + (1 - z_{12}^3)\mathbf{U}_3 + \lambda\mathbf{U}_6. \quad (35)$$

It follows that the image of \mathbf{U}_2 is a surface S in $S^5(1)$, and we now show that this is a minimal surface.

LEMMA 2. *The vectors $X = d\mathbf{U}_2(\mathbf{e}_2)$ and $Y = d\mathbf{U}_2(\mathbf{e}_3)$ are perpendicular and have the same (non-zero) length.*

THEOREM 2. *The image S of \mathbf{U}_2 is a minimal surface in $S^5(1)$.*

Proof. Let II denote the second fundamental form of S in $S^5(1)$. It follows from Lemma 2 that we need only check that $II(X, X) + II(Y, Y) = 0$, or, equivalently, that $dX(\mathbf{e}_2) + dY(\mathbf{e}_3)$ is a linear combination of \mathbf{U}_2 , X and Y . In fact, a calculation using (14), (15), (23) and (33) shows that the component of $dX(\mathbf{e}_2) + dY(\mathbf{e}_3)$ perpendicular to \mathbf{U}_2 is equal to $-z_{32}^3X + z_{22}^3Y$, from which the result follows. \square

We now investigate the *ellipse of curvature* E of S . Recall that the ellipse of curvature at a point p of a minimal surface is that (possibly degenerate) ellipse in the first normal space given by

$$E = \{II(Z, Z) \mid Z \text{ is a unit tangent vector to } S \text{ at } p\}.$$

LEMMA 3. *The ellipse of curvature at any point of S is not a circle. The direction of the minor axis is given by $II(X, X)$ and that of the major axis by $II(X, Y)$.*

Proof. We first note that $2II(X, X)$ (resp. $2II(X, Y)$) is equal to the component of $dX(\mathbf{e}_2) - dY(\mathbf{e}_3)$ (resp. $dX(\mathbf{e}_3) + dY(\mathbf{e}_2)$) perpendicular to S . In order to facilitate the calculations, which we carried out using Mathematica, we let

$$\mathbf{K}_1 = dX(\mathbf{e}_2) - dY(\mathbf{e}_3) + 3(z_{22}^3Y + z_{32}^3X),$$

and

$$\mathbf{K}_2 = dX(\mathbf{e}_3) + dY(\mathbf{e}_2) + 3(z_{32}^3Y - z_{22}^3X),$$

so that $2II(X, X)$ and $2II(X, Y)$ are the components of \mathbf{K}_1 and \mathbf{K}_2 perpendicular to S . A calculation shows that

$$\mathbf{K}_1 = \mu_2\mathbf{U}_1 + \mu_1\mathbf{U}_3 + \mu_3\mathbf{U}_5 + \mu_4\mathbf{U}_6, \quad (36)$$

$$\mathbf{K}_2 = \mu_1\mathbf{U}_1 - \mu_2\mathbf{U}_3 - 4\lambda\mathbf{U}_4 + \mu_4\mathbf{U}_5 - \mu_3\mathbf{U}_6, \quad (37)$$

where

$$\mu_1 = 4(z_{22}^3 - z_{12}^3z_{22}^3 + z_{21}^2z_{32}^3) + \mathbf{e}_3(z_{12}^3) + \mathbf{e}_2(z_{21}^2), \quad (38)$$

$$\mu_2 = 4(z_{21}^2z_{22}^3 - z_{32}^3 + z_{12}^3z_{32}^3) + \mathbf{e}_2(z_{12}^3) - \mathbf{e}_3(z_{21}^2), \quad (39)$$

$$\mu_3 = 2(b(1 - z_{12}^3) - az_{21}^2), \quad (40)$$

$$\mu_4 = 2(a(z_{12}^3 - 1) - bz_{21}^2). \quad (41)$$

It is clear from (36) and (37) that \mathbf{K}_1 and \mathbf{K}_2 are orthogonal vectors, and from (34) and (35) that

$$(\mathbf{K}_1, X) = (\mathbf{K}_2, Y) \quad \text{and} \quad (\mathbf{K}_1, Y) = -(\mathbf{K}_2, X),$$

where (\cdot, \cdot) denotes the standard inner product on \mathbb{R}^6 . Hence the components of \mathbf{K}_1 and \mathbf{K}_2 tangential to S are orthogonal and have the same length. It now follows that $II(X, X)$ and $II(X, Y)$ are also orthogonal, so that $II(X, X)$ and $II(X, Y)$ lie along the axes of the ellipse of curvature. Also, it is clear from (36) and (37) that $|\mathbf{K}_2|^2 - |\mathbf{K}_1|^2 = 16\lambda^2$, implying that

$$|II(X, Y)|^2 - |II(X, X)|^2 = 4\lambda^2. \quad (42)$$

Hence the ellipse of curvature is not a circle since $\lambda \neq 0$. \square

We note from (42) that there is a positive function ϕ such that

$$|II(X, Y)| = 2\lambda \cosh \phi \quad \text{and} \quad |II(X, X)| = 2\lambda \sinh \phi. \quad (43)$$

We may express the eccentricity e of the ellipse of curvature in terms of ϕ . In fact,

$$\begin{aligned} e &= \sqrt{1 - \frac{|II(X, X)|^2}{|II(X, Y)|^2}} \\ &= \operatorname{sech} \phi. \end{aligned}$$

We have seen that X and Y determine geometrically significant directions on S , so we would therefore expect that $dX(\mathbf{e}_1)$ and $dY(\mathbf{e}_1)$ are scalar multiples of X and Y respectively. In fact, it follows from (21), (13) and (33) that $dX(\mathbf{e}_1) = -z_{21}^2 X$ and $dY(\mathbf{e}_1) = -z_{21}^2 Y$.

We now determine conditions on M in order that S lies in a totally geodesic $S^3(1)$ in $S^5(1)$.

THEOREM 3. *Let S be the minimal surface in $S^5(1)$ determined by \mathbf{U}_2 . Then the following conditions are equivalent.*

- (i) *The surface S is contained in a totally geodesic $S^3(1)$ in $S^5(1)$.*
- (ii) *The ellipse of curvature of S is degenerate at each point of S .*
- (iii) *The vector field \mathbf{e}_1 on M is a Killing vector field.*
- (iv) $\langle \nabla_{\mathbf{e}_2} \mathbf{e}_1, \mathbf{e}_2 \rangle = 0$.
- (v) $\langle \nabla_{\mathbf{e}_3} \mathbf{e}_1, \mathbf{e}_3 \rangle = 0$.

REMARK. Minimal Lagrangian submanifolds M admitting a unit length Killing vector field whose integral curves are geodesics in $\mathbb{C}P^3(4)$ are investigated in [6]. In particular, explicit examples of minimal Lagrangian tori admitting such a vector field are constructed.

Proof. The equivalence of (iv) and (v) is immediate from (12).

Minimality of S , together with the Codazzi equations for S show that (i) holds if and only if $II(X, X) \equiv 0$ on an open subset of S or, equivalently, (ii) holds.

On the other hand, (iii) holds if and only if \mathbf{e}_1 satisfies the Killing equations, namely $\langle \nabla_U \mathbf{e}_1, V \rangle + \langle \nabla_V \mathbf{e}_1, U \rangle = 0$ for all vectors U, V tangential to M . It follows from (12) that this holds if and only if (iv) holds.

Hence, we may prove the theorem by showing that the vector \mathbf{K}_1 given by (36) is a linear combination of X and Y if and only if $z_{21}^2 = 0$.

We first assume that $z_{21}^2 = 0$. In this case, using (23) we see that

$$\mathbf{K}_1 = 2(2z_{32}^3(z_{12}^3 - 1) + b\lambda)\mathbf{U}_1 - 2(2z_{22}^3(z_{12}^3 - 1) + a\lambda)\mathbf{U}_3 + 2b(1 - z_{12}^3)\mathbf{U}_5 + 2a(z_{12}^3 - 1)\mathbf{U}_6.$$

This is a linear combination of X and Y if and only if both the following equations hold.

$$a(z_{12}^3 - 1)^2 = 2z_{22}^3\lambda(z_{12}^3 - 1) + a\lambda^2, \quad (44)$$

and

$$b(z_{12}^3 - 1)^2 = 2z_{32}^3\lambda(z_{12}^3 - 1) + b\lambda^2. \quad (45)$$

Using (21), these equations simplify to

$$az_{12}^3 = \lambda z_{22}^3, \quad (46)$$

and

$$bz_{12}^3 = \lambda z_{32}^3. \quad (47)$$

However, it follows from (12) that $[\mathbf{e}_1, \mathbf{e}_2] = 0$, and, applying this to z_{12}^3 using (13), (17), (21) and (23), we obtain (46). Similarly, $[\mathbf{e}_1, \mathbf{e}_3] = 0$, and, applying this to z_{12}^3 , we obtain (47). Thus \mathbf{K}_1 is a linear combination of X and Y as required.

Conversely, assume that \mathbf{K}_1 is a linear combination of X and Y . It then follows from (34), (35) and (36) that

$$\lambda\mu_1 = -\mu_3 z_{21}^2 + \mu_4(1 - z_{12}^3), \quad (48)$$

and

$$\lambda\mu_2 = \mu_3(1 - z_{12}^3) + \mu_4 z_{21}^2. \quad (49)$$

We may use the above two equations, together with (23) to obtain the following algebraic expressions for $\mathbf{e}_2(z_{21}^2)$, $\mathbf{e}_3(z_{21}^2)$, $\mathbf{e}_2(z_{12}^3)$ and $\mathbf{e}_3(z_{12}^3)$.

$$\lambda\mathbf{e}_2(z_{21}^2) = -(z_{12}^3 - 1)^2 a + 2(z_{12}^3 - 1)(bz_{21}^2 + \lambda z_{22}^3) + a((z_{21}^2)^2 + \lambda^2) - 2\lambda z_{21}^2 z_{32}^3, \quad (50)$$

$$\lambda\mathbf{e}_3(z_{21}^2) = -(z_{12}^3 - 1)^2 b + 2(z_{12}^3 - 1)(-az_{21}^2 + \lambda z_{32}^3) + b((z_{21}^2)^2 + \lambda^2) + 2\lambda z_{21}^2 z_{22}^3, \quad (51)$$

$$\lambda\mathbf{e}_2(z_{12}^3) = (z_{12}^3 - 1)^2 b + 2(z_{12}^3 - 1)(az_{21}^2 - \lambda z_{32}^3) + b(-(z_{21}^2)^2 + \lambda^2) - 2\lambda z_{21}^2 z_{22}^3, \quad (52)$$

$$\lambda\mathbf{e}_3(z_{12}^3) = -(z_{12}^3 - 1)^2 a + 2(z_{12}^3 - 1)(bz_{21}^2 + \lambda z_{22}^3) + a((z_{21}^2)^2 - \lambda^2) - 2\lambda z_{21}^2 z_{32}^3. \quad (53)$$

We now consider the integrability conditions for λ . In fact, the only one we will need is obtained by applying $\nabla_{\mathbf{e}_2}\mathbf{e}_3 - \nabla_{\mathbf{e}_3}\mathbf{e}_2 - [\mathbf{e}_2, \mathbf{e}_3]$ to λ and equating the answer to zero. Carrying out this process using (12)-(15), we obtain

$$\mathbf{e}_2(z_{22}^3) + \mathbf{e}_3(z_{32}^3) = 2z_{12}^3 z_{21}^2. \quad (54)$$

Using the above equations and (12)-(19), (21)-(24), a calculation (for which we used Mathematica) shows that the integrability condition obtained by applying $\nabla_{\mathbf{e}_2}\mathbf{e}_3 - \nabla_{\mathbf{e}_3}\mathbf{e}_2 - [\mathbf{e}_2, \mathbf{e}_3]$ to z_{12}^3 and equating the answer to zero reduces to $z_{21}^2 = 0$. This completes the proof of the theorem. \square

We now return to the general situation governed by the ruled minimal Lagrangian submanifold M of $\mathbb{C}P^3(4)$. We note that the arguments applied to \mathbf{U}_2 may be used to show that the image \hat{S} of \mathbf{U}_4 is also a minimal surface in $S^5(1)$. We now investigate the relation between the two minimal surfaces S and \hat{S} .

LEMMA 4. *If S is contained in a totally geodesic $S^3(1)$ then \hat{S} is the polar of S in the sense of Lawson [9].*

Proof. In this situation, $II(X, X) = 0$, so that \mathbf{K}_1 is a linear combination of X and Y . We also have from Theorem 3 that $z_{21}^2 = 0$, so it follows from (34), (35), (36) and (37) that $2II(X, Y)$, the component of \mathbf{K}_2 perpendicular to X and Y , is equal to $-4\lambda\mathbf{U}_4$. In particular, \hat{S} is in the totally geodesic $S^3(1)$ containing S and at each point is orthogonal to S and the tangent space to S . Thus \hat{S} is the polar of S . \square

We now assume that S is not contained in a totally geodesic $S^3(1)$. Theorem 3, together with real analyticity of minimal surfaces imply that, by restricting to an open dense subset of M , we may assume that z_{21}^2 is a nowhere vanishing function. Therefore, by replacing \mathbf{e}_1 with $-\mathbf{e}_1$ (and interchanging \mathbf{e}_2 and \mathbf{e}_3 in order to keep λ positive) if necessary, we may also assume that z_{21}^2 is a strictly positive function. We now let \mathbf{N} be the unit vector in \mathbb{R}^6 such that

$$\{\mathbf{U}_2, X, Y, II(X, X), II(X, Y), \mathbf{N}\} \text{ is a positively oriented orthogonal frame of } \mathbb{R}^6. \quad (55)$$

It follows from (34), (35) and (36) that \mathbf{U}_4 is orthogonal to \mathbf{U}_2, X, Y , and $II(X, X)$, and so is a linear combination of $II(X, Y)$ and \mathbf{N} . Also, it follows from (37) that $(\mathbf{U}_4, II(X, Y)) = -2\lambda$ so that, using the positive function ϕ introduced in (43), we see that

$$\mathbf{U}_4 = -\operatorname{sech} \phi \frac{II(X, Y)}{|II(X, Y)|} + \epsilon \tanh \phi \mathbf{N},$$

where $\epsilon = \pm 1$.

In order to determine the sign of ϵ , we compute the determinant of $(\mathbf{U}_2, X, Y, II(X, X), II(X, Y), \mathbf{U}_4)$. In fact,

$$\begin{aligned} 4 \det(\mathbf{U}_2, X, Y, II(X, X), II(X, Y), \mathbf{U}_4) &= \det(\mathbf{U}_2, X, Y, \mathbf{K}_1, \mathbf{K}_2, \mathbf{U}_4) \\ &= -(\mu_1^2 + \mu_2^2)\lambda^2 + 2((z_{12}^3 - 1)(\mu_1\mu_4 + \mu_2\mu_3) + z_{21}^2(\mu_1\mu_3 - \mu_2\mu_4))\lambda \\ &\quad - (\mu_3^2 + \mu_4^2)((z_{12}^3 - 1)^2 + (z_{21}^2)^2). \end{aligned}$$

Regarding this as a quadratic equation in λ , we see that its discriminant is given by

$$-((\mu_4\mu_2 + \mu_3\mu_1)(z_{12}^3 - 1) + z_{21}^2(\mu_2\mu_3 - \mu_1\mu_4))^2.$$

This is always non-positive, implying that

$$\det(\mathbf{U}_2, X, Y, II(X, X), II(X, Y), \mathbf{U}_4) \leq 0,$$

from which it follows that $\epsilon = -1$.

We remark that the minimal surface \hat{S} determined by \mathbf{U}_4 may be determined directly from S together with a choice of direction along the major axis of the ellipse of curvature E of S . In fact, the unit vector \mathbf{N} determined by (55) does not depend on the choice of basis $\{X, Y\}$ of the tangent space of S , so if V is the unit vector in

the direction determined by $II(X, Y)$ along the major axis of E then we may define $\hat{\cdot}: S \rightarrow \hat{S}$, where

$$\hat{p} = (-\operatorname{sech} \phi V - \tanh \phi \mathbf{N})(p), \quad p \in S.$$

We will call this the $(-)$ transform of S . This, together with a related construction called the $(+)$ transform, may be described geometrically as follows. Let E be the ellipse of curvature of S at a point $p \in S$ and let P be the 3-plane orthogonal to \mathbf{U}_2 and its tangent space. Let R_θ be the rotation of P about the minor axis of E through an angle θ , $0 \leq \theta \leq \pi/2$, such that $R_\theta(V)$ makes an acute angle with \mathbf{N} and having the property that the orthogonal projection of $R_\theta(E)$ onto the plane containing E is a circle. Then the inverse rotation $R_{-\theta}$ has a similar geometric effect on E , and we define the $(+)$ transform and $(-)$ transform of S by setting

$$p^+ = R_\theta(\mathbf{N}) = (-\operatorname{sech} \phi V + \tanh \phi \mathbf{N})(p), \quad p \in S, \quad (56)$$

and

$$p^- = R_{-\theta}(-\mathbf{N}) = (-\operatorname{sech} \phi V - \tanh \phi \mathbf{N})(p), \quad p \in S. \quad (57)$$

Thus \mathbf{U}_4 is obtained by applying the $(-)$ transform to the minimal surface determined by \mathbf{U}_2 , and we now show that \mathbf{U}_2 is obtained by applying the $(+)$ transform to the minimal surface determined by \mathbf{U}_4 . We begin by noting that if, in our construction, we replace \mathbf{e}_3 by $-\mathbf{e}_3$ then a suitable lift to $SU(4)$ gives the map $\tilde{\mathbf{U}} = (\tilde{\mathbf{U}}_1, \dots, \tilde{\mathbf{U}}_6): M \rightarrow SO(6)$ where

$$\tilde{\mathbf{U}}_1 = \mathbf{U}_6, \quad \tilde{\mathbf{U}}_2 = -\mathbf{U}_4, \quad \tilde{\mathbf{U}}_3 = -\mathbf{U}_5, \quad \tilde{\mathbf{U}}_4 = \mathbf{U}_2, \quad \tilde{\mathbf{U}}_5 = \mathbf{U}_3, \quad \tilde{\mathbf{U}}_6 = -\mathbf{U}_1. \quad (58)$$

Thus, from Theorem 3, \hat{S} is not contained in a totally geodesic $S^3(1)$. Also,

$$\tilde{\mathbf{U}}_4 = -\operatorname{sech} \tilde{\phi} \frac{\tilde{II}(\tilde{X}, \tilde{Y})}{|\tilde{II}(\tilde{X}, \tilde{Y})|} - \tanh \tilde{\phi} \tilde{\mathbf{N}}, \quad (59)$$

where $\tilde{\phi} > 0$ is such that $\operatorname{sech} \tilde{\phi}$ is the eccentricity of the ellipse of curvature of the surface \tilde{S} determined by $\tilde{\mathbf{U}}_2$, $\tilde{X} = d\tilde{\mathbf{U}}_2(\mathbf{e}_2)$, $\tilde{Y} = d\tilde{\mathbf{U}}_2(-\mathbf{e}_3)$, \tilde{II} is the second fundamental form of \tilde{S} , and $\tilde{\mathbf{N}}$ is the unit vector in \mathbb{R}^6 such that $\{\tilde{\mathbf{U}}_2, \tilde{X}, \tilde{Y}, \tilde{II}(\tilde{X}, \tilde{X}), \tilde{II}(\tilde{X}, \tilde{Y}), \tilde{\mathbf{N}}\}$ is a positively oriented orthogonal frame of \mathbb{R}^6 .

Now let $\hat{X} = d\mathbf{U}_4(\mathbf{e}_2)$, $\hat{Y} = d\mathbf{U}_4(\mathbf{e}_3)$, and let \hat{II} be the second fundamental form of \hat{S} . Then $\hat{II}(\hat{X}, \hat{X}) = -\tilde{II}(\tilde{X}, \tilde{X})$ is along the minor axis of the ellipse of curvature of \hat{S} , while $\hat{II}(\hat{X}, \hat{Y}) = \tilde{II}(\tilde{X}, \tilde{Y})$ is along the major axis. Now let $\hat{\mathbf{N}}$ be the unit vector in \mathbb{R}^6 such that $\{\mathbf{U}_4, \hat{X}, \hat{Y}, \hat{II}(\hat{X}, \hat{X}), \hat{II}(\hat{X}, \hat{Y}), \hat{\mathbf{N}}\}$ is a positively oriented orthogonal basis of \mathbb{R}^6 . It then follows from (58) that $\tilde{\mathbf{N}} = -\hat{\mathbf{N}}$, and from (58) and (59) that if $\hat{\phi} > 0$ is such that $\operatorname{sech} \hat{\phi}$ is the eccentricity of the ellipse of curvature of \hat{S} , then

$$\mathbf{U}_2 = -\operatorname{sech} \hat{\phi} \frac{\hat{II}(\hat{X}, \hat{Y})}{|\hat{II}(\hat{X}, \hat{Y})|} + \tanh \hat{\phi} \hat{\mathbf{N}}.$$

Thus, taking the direction along the major axis of the ellipse of curvature of \hat{S} to be that determined by $\hat{II}(\hat{X}, \hat{Y})$, then applying the $(+)$ transform to \hat{S} gives us S .

The following theorem summarises the results of the paper.

THEOREM 4. *A ruled minimal Lagrangian submanifold M of $\mathbb{CP}^3(4)$ defines two minimal surfaces S and \hat{S} in $S^5(1)$. These surfaces are related geometrically in that \hat{S} is obtained from S by the $(-)$ transform and S is obtained from \hat{S} by the $(+)$ transform.*

Thus, ruled minimal Lagrangian submanifolds of $\mathbb{C}P^3(4)$ induce two constructions on what could be a special class of minimal surfaces in $S^5(1)$, namely a $(-)$ transform, given by (57), producing \hat{S} from S and a $(+)$ transform, given by (56), producing S from \hat{S} .

In a forthcoming paper [4] we shall show that if we apply either of these constructions to an arbitrary minimal surface with non-circular non-degenerate ellipse of curvature in $S^5(1)$ then we obtain another minimal surface in $S^5(1)$. As a consequence of this we will show that every such minimal surface in $S^5(1)$ may be constructed locally in the manner described in the present paper from a ruled minimal Lagrangian submanifold of $\mathbb{C}P^3(4)$.

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